18.152 PROBLEM SET 3 SOLUTIONS

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1. Problem 3

Most students worked this out. The idea is to use the polar coordinate and the separation of variables.

Proof. Using the polar coordinate (r, θ) on $\mathbb{R}^2$, we have

$$\Delta u = \partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u$$

for $r \in [1, \infty)$ and $\theta \in [0, 2\theta)$. We can do the separation of variables at this step, but let us make another change of variable to simplify the problem:

$$\rho = \log r \in [0, \infty).$$

so by the chain rule,

$$\partial_\rho = \frac{\partial r(\rho)}{\partial \rho} \cdot \partial_r = r \cdot \partial_r,$$

and

$$\partial_\rho^2 = r \cdot \partial_r(r \cdot \partial_r) = r^2 \partial_r^2 + r \cdot \partial_r = r^2(\partial_r^2 + \frac{1}{r} \cdot \partial_r).$$

As a result,

$$\frac{1}{r^2}(\partial_\rho^2 + \partial_\theta^2)u = \partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u = 0,$$

where $u : [0, \infty)_\rho \times [0, 2\pi)_\theta \to \mathbb{R}$. We formally write

$$u(\rho, \theta) = \sum_{n \geqslant 0} a_n(\rho)\cos(n\theta) + \sum_{n \geqslant 1} b_n(\rho)\sin(n\theta),$$

where $a_n, b_n : [0, \infty) \to \mathbb{R}$. Notice that each a_n (and b_n) is a constant function on each circle $\{\rho\} \times [0, 2\pi)$. Since u is a harmonic function,

$$(\partial_\rho^2 + \partial_\theta^2)a_n(\rho)\cos(n\theta) = [\partial_\rho^2 a_n(\rho) - n^2 a_n(\rho)]\cos(n\theta).$$

As a result,

$$\begin{cases} \partial_\rho^2 a_n(\rho) - n^2 a_n(\rho) = 0 & n \geqslant 0, \\ \partial_\rho^2 b_n(\rho) - n^2 b_n(\rho) = 0 & n \geqslant 1, \end{cases}$$

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so
$$\begin{cases} a_0 &= a_0^1 + a_0^2\rho, \\ a_n &= a_n^1 e^{-n\rho} + a_n^2 e^{n\rho} \quad n \geqslant 1, \\ b_n &= b_n^1 e^{-n\rho} + b_n^2 e^{n\rho} \quad n \geqslant 1. \end{cases}$$
where a_n^1, a_n^2 and b_n^1, b_n^2 are some constant numbers. So
$$u(\rho,\theta) = a_0^1 + a_0^2\rho + \sum_{n\geqslant 1}(a_n^1 e^{-n\rho} + a_n^2 e^{n\rho})\cos(n\theta) + \sum_{n\geqslant 1}(b_n^1 e^{-n\rho} + b_n^2 e^{n\rho})\sin(n\theta).$$
The boundary condition of u then says
$$u(0,\theta) = \cos 2\theta, \ \lim_{\rho\to\infty} e^{-\rho}u(\rho,\theta) = 0.$$
The second condition implies that the coefficient of $e^{n\rho}$ must vanish if $n \geqslant 1$, so
$$u(\rho,\theta) = a_0^1 + a_0^2\rho + \sum_{n\geqslant 1} a_n^1 e^{-n\rho}\cos(n\theta) + \sum_{n\geqslant 1} b_n^1 e^{-n\rho}\sin(n\theta).$$
By the first condition,
$$\cos 2\theta = u(0,\theta) = a_0^1 + \sum_{n\geqslant 1} a_n^1 \cos(n\theta) + \sum_{n\geqslant 1} b_n^1 \sin(n\theta),$$
so $a_2^1 = 2$ and all other terms vanish. In conclusion,
$$u(\rho,\theta) = a_0^2\rho + \cos(2\theta)$$
Using the coordinates (r,θ),
$$u(r,\theta) = a_0^2 \log r + \cos(2\theta).$$
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